ON A CLASS OF DUAL INTEGRAL EQUATIONS AND THEIR APPLICATIONS TO THE THEORY OF ELASTICITY

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The paper studies dual integral equations connected with the generalized Mehler-Fock transform by means of associated spherical functions

$$P^m_{-1/2+i\tau}(\cosh\alpha)$$

By a method similar to that developed in [1 and 2] the solution can in general be expressed in terms of one unknown function which satisfies a regular Fredholm equation. Certain classes of boundary-value problems in mathematical physics and the theory of elasticity with mixed boundary conditions are indicated which can be solved by the method developed in the paper.

As a specific example the paper solves the problem of an elastic half-space twisted by a hollow cylindrical die.

1. Consider dual integral equations of the form

$$\int_0^\infty A(\tau)\, P^m_{-1/2+i\tau}(\cosh\alpha)\,[1+g(\tau)]\,d\tau = f(\alpha) \qquad (0\leqslant\alpha<\alpha_0) \tag{1.1}$$

$$\int_0^\infty \tau A(\tau)\, P^m_{-1/2+i\tau}(\cosh\alpha)\tanh\pi\tau\,d\tau = 0 \qquad (\alpha_0<\alpha<\infty) \tag{1.2}$$

Here $A(\tau)$ is the function sought and $g(\tau)$ and $f(\alpha)$ are known functions. ~ $(\cosh\alpha)$ are associated spherical functions $(m = 0, 1, 2, \ldots)$.

By introducing an auxiliary function $\varphi(t)$ given by

$$A(\tau) = \int_0^{\alpha_0} \varphi(t)\cos\tau t\,dt \tag{1.3}$$

and making use of Formulas [3]

$$P_\nu^m(\lambda) = (\lambda^2-1)^{1/2 m}\frac{d^m P_\nu(\lambda)}{d\lambda^m} \qquad (\lambda=\cosh\alpha) \tag{1.4}$$

$$\int_0^\infty \tanh \pi\tau P_{-1/2+i\tau}(\cosh\alpha) \sin \tau t \, d\tau = \begin{cases} 0, & t < \alpha \\ [2(\cosh t - \cosh\alpha)]^{-1/2}, & t > \alpha \end{cases} \tag{1.5}$$

it can be shown that Equation (1.2) is satisfied identically for any function $\varphi(t)$ which has a continuous derivative. If we now integrate (1.1) m times with respect to λ, we reduce this equation to the form

$$\int_0^\infty A(\tau)[1 + g(\tau)] P_{-1/2+i\tau}(\cosh\alpha) \, d\tau = F(\alpha) + \sum_{k=0}^{m-1} c_k \lambda^k = \chi(\alpha) \tag{1.6}$$

where c_k are constants and (*)

$$F(\alpha) = \int_1^\lambda \int_1^\lambda \cdots \int_1^\lambda f(\alpha)(\lambda^2 - 1)^{-1/2m} \, d\lambda^m \tag{1.7}$$

By substituting (1.3) into (1.6) and using Formula [3]

$$\int_0^\infty P_{-1/2+i\tau}(\cosh\alpha) \cos \tau t \, d\tau = \begin{cases} [2(\cosh\alpha - \cosh t)]^{-1/2}, & t < \alpha \\ 0, & t > \alpha \end{cases} \tag{1.8}$$

we transform (1.6) into an Abel integral equation

$$\int_0^\alpha \Phi(x) [2(\cosh\alpha - \cosh x)]^{-1/2} dx = \chi(\alpha) \tag{1.9}$$

Here

$$\varphi(x) + \frac{1}{\pi} \int_0^{\alpha_0} [G(t + x) + G(t - x)] \varphi(t) \, dt = \Phi(x)$$

$$G(y) = \int_0^\infty g(\tau) \cos \tau y \, d\tau \tag{1.10}$$

Since the solution of (1.9) is known,

$$\Phi_0(x) = \frac{2}{\pi} \frac{d}{dx} \int_0^x [2(\cosh x - \cosh\alpha)]^{-1/2} \chi(\alpha) \sinh \alpha \, d\alpha \tag{1.11}$$

the problem resolves itself into one of determining $\varphi(x)$ from the integral equation (1.10) with a continuous symmetric kernel.

2. The solution obtained contains m arbitrary constants which can be found by imposing certain supplementary conditions. Proceeding from the formulation of the boundary-value problems which lead to the dual integral equations under discussion (Section 3) and, following the same procedure as in [1 and 4], we make the requirement that the function

$$\psi(\alpha) = \int_0^\infty \tau A(\tau) \tanh \pi\tau P^m_{-1/2+i\tau}(\cosh\alpha) \, d\tau \tag{2.1}$$

*) Here and in what follows it is assumed that the given functions satisfy certain conditions which ensure the convergence of the integrals and allow the order of integration to be reversed, etc.

is integrable within the range $0 \le \alpha \le \alpha_0$. Substituting (1.3) into (2.1) and applying (1.4) and (1.5), we can reduce (2.1) to the form

$$\psi(\alpha)\big|_{0\leqslant\alpha<\alpha_0} = (\lambda^2-1)^{1/2 m}\frac{d^m}{d\lambda^m}\left\{\varphi(\alpha_0)\,[2(\cosh\alpha_0-\cosh\alpha)]^{-1/2}-\right.$$

$$\left.-\int_\alpha^{\alpha_0}\varphi'(t)\,[2(\cosh t-\cosh\alpha)]^{-1/2}\,dt\right\} \tag{2.2}$$

Carrying out the successive integrations by parts, we can show that the above requirement is equivalent to the condition that

$$\varphi^{(i)}(\alpha_0)=0 \qquad (i=0,1,\ldots,m-1) \tag{2.3}$$

As $\alpha \to \alpha_0 - 0$ the function $\psi(\alpha)$ assumes the order of $(\cosh\alpha_0 - \cosh\alpha)^{1/2}$.

It can easily be seen that conditions (2.3) determine the constants c_k. In fact, if we express $\varphi(x)$ in the form of the sum

$$\varphi(x)=\varphi_1(x)+\varphi_2(x), \qquad \varphi_2(x)=\sum_{k=0}^{m-1}c_k\omega_k(x) \tag{2.4}$$

and set $\Phi(x) = \Phi_1(x) + \Phi_2(x)$, where

$$\Phi_1(x)=\frac{2}{\pi}\frac{d}{dx}\int_0^x[2(\cosh x-\cosh\alpha)]^{-1/2}F(\alpha)\sinh\alpha\,d\alpha \tag{2.5}$$

$$\Phi_2(x)=\sum_{k=0}^{m-1}c_k\frac{dI_k}{dx}, \qquad I_k(x)=\frac{2}{\pi}\int_0^x[2(\cosh x-\cosh\alpha)]^{-1/2}\cosh^k\alpha\sinh\alpha\,d\alpha \tag{2.6}$$

we obtain from (1.10) for the required functions $\varphi_1(x)$ and $\omega_k(x)$ different Fredholm equations with given right-hand sides $\Phi_1(x)$ and dI_k/dx . Conditions (2.3) now become the linear algebraic system

$$\varphi_1^{(i)}(\alpha_0)+\sum_{k=0}^{m-1}c_k\omega_k^{(i)}(\alpha_0)=0 \qquad (i=0,1,\ldots,m-1) \tag{2.7}$$

in the required quantities c_k .

If in Expression (2.6) for $I_k(x)$ we make the substitution $\sinh\frac{1}{2}\alpha = \sinh\frac{1}{2}x\sin\theta$ and introduce, the polynomials $P_{2k}(\sinh\frac{1}{2}\alpha) = \cosh^k\alpha$, we can express the integrals $I_k(x)$ in the form

$$I_k(x)=\frac{4}{\pi}\sinh{}^1/_2x\int_0^{1/2\pi}P_{2k}(\sinh{}^1/_2x\sin\theta)\sin\theta\,d\theta \tag{2.8}$$

It follows that they are known polynomials of order $2k + 1$ in $\sinh\frac{1}{2}x$

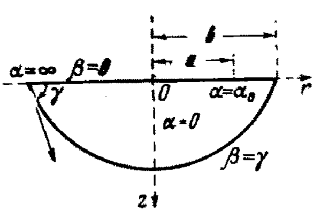

Fig. 1

3. Certain classes of boundary-value problems in potential theory and in the theory of elasticity with mixed boundary conditions reduce to dual integral equations of the form (1.1) and (1.2). As an example of one application we shall con- consider the boundary-value problem for a spherical segment (Fig.1) when the required harmonic function $u(r, \theta, z)$ vanishes on the spherical surface and satisfies the mixed conditions

$$u = f(r,\theta), \quad 0 \leqslant r < a; \qquad \partial u / \partial z = 0, \quad a < r < b \tag{3.1}$$

on the flat surface $z = 0$.

If we introduce toroidal coordinates by means of Formulas [5]

$$x = \frac{b \sinh\alpha \cos\theta}{\cosh\alpha + \cos\beta}, \quad y = \frac{b \sinh\alpha \sin\theta}{\cosh\alpha + \cos\beta}, \quad z = \frac{b \sin\beta}{\cosh\alpha + \cos\beta} \quad \begin{pmatrix} 0 \leqslant \alpha < \infty \\ 0 \leqslant \beta \leqslant \gamma \end{pmatrix} \tag{3.2}$$

then the equation of the spherical surface of the segment will be $\beta = \gamma$. On the flat surface ($\beta = 0$) the line of division between boundary conditions of the first and second kind is defined by the circle $\alpha = \alpha_0$ ($\tanh {}^1/_2\alpha_0 = b/a$), and at $z = 0$, $r \to b$ we have that $\alpha \to \infty$.

We shall seek a solution to the problem which satisfies the condition $u|_{\beta=\gamma} = 0$, in the form (*)

$$u = \sqrt{\cosh\alpha + \cos\beta} \sum_{m=-\infty}^{\infty} e^{im\theta} \int_0^\infty B_m(\tau) \frac{\sinh(\gamma - \beta)\tau}{\sinh \gamma\tau} P^m_{-1/2+i\tau}(\cosh\alpha)\, d\tau \tag{3.3}$$

If we apply the conditions (3.1) and then expand the given function f in a Fourier series in the angular coordinate

$$f = \sqrt{\cosh\alpha + 1} \sum_{m=-\infty}^{\infty} f_m(\alpha) e^{im\theta} \tag{3.4}$$

we arrive at the dual equations

$$\int_0^\infty B_m(\tau) P^m_{-1/2+i\tau}(\cosh\alpha)\, d\alpha = f_m(\alpha) \qquad (0 \leqslant \alpha < \alpha_0) \tag{3.5}$$

$$\int_0^\infty \tau B_m(\tau) \coth \gamma\tau P^m_{-1/2+i\tau}(\cosh\alpha)\, d\tau = 0 \qquad (\alpha_0 < \alpha < \infty) \tag{3.6}$$

which, by means of the substitution $B_m \coth \gamma\tau = A \tanh \pi\tau$ reduce to the form (1.1) and (1.2). Thus, we have reduced the problem to a Fredholm equation in which

$$g(\tau) = -\frac{\cosh(\pi - \gamma)\tau}{\cosh \pi\tau \cosh\gamma\tau}$$

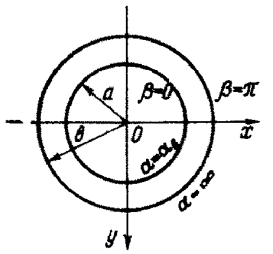

Fig. 2

Note that in the case when all the boundary conditions are nonhomogeneous the problem can be reduced to the dual eqaations (1.1) and (1,2) by means of the generalized Mehler-Fock integral transformation (Section 4).

Problems of this sort are encountered, for example, in the study of stationary processes in the theory of heat conduction. Dual equations

*) See, for example, [6]. Note that

$$P^{-m}_{-1/2+i\tau}(\cosh\alpha) = \frac{\Gamma(1/2 + i\tau - m)}{\Gamma(1/2 + i\tau + m)} P^m_{-1/2+i\tau}(\cosh\alpha)$$

of the type (3.5) and (3.6) with $m = 1$ arise in the problem of the torsion of a truncated sphere [1]. It will become clear in similar problems what is meant by the supplementary condition used in Section 2 which, obviously, amounts to the requirement that the normal derivative $\partial u/\partial z$ is integrable over the area $z = 0$, $0 \leq r \leq a$. It can be shown that the solution to the boundary-value problem in this case is unique. In physical problems this requirement is equivalent to the condition that the flow of heat is finite over the surface $z = 0$, $0 \leq r \leq a$, or to the condition that the torque is finite, etc.

In the particular case when $\gamma = \pi$ we arrive at a mixed problem for a half-space with a doubly connected ($r = a$ and $r = b$ Fig.2) dividing line between boundary conditions of the first and second kind (*). It has been shown [6] that statical problems of the theory of elasticity for a half-space reduce to such problems when the normal stress is given on the flat ring ($z = 0$, $a < r < b$) and when the normal displacement is specified on the remainder of the boundary (the shear stresses are considered to be known over the whole of the surface $z = 0$).

In particular we can solve problems on the deformation of an infinite body with a plane annular slit [7]. A further example of this class of problem is the electrostatic problem of the field of a circular disk situated in the opening of a diaphragm.

4. We can apply the method described above to boundary-value problems for a spherical segment when over the area $z = 0$, $0 \leq r < a$ a homogeneous boundary condition of the second kind is specified and for $z = 0$, $a < r < b$ the value of the required function u is given. For the condition that $u|_{\beta=\gamma} = 0$ the solution retains the form (3.3) and reduces to the dual equations

$$\int_0^\infty \tau B(\tau) \coth \gamma\tau P^m_{-1/2+i\tau}(\cosh\alpha)\, d\tau = 0 \qquad (0 \leqslant \alpha < \alpha_0)$$

$$\int_0^\infty B(\tau) P^m_{-1/2+i\tau}(\cosh\alpha)\, d\tau = f(\alpha) \qquad (\alpha_0 < \alpha < \infty) \tag{4.1}$$

Expanding $f(\alpha)$ in the form of a generalized Mehler-Fock integral [6]

$$f(\alpha) = (-1)^m \int_0^\infty \tau \tanh \pi\tau P^m_{-1/2+i\tau}(\cosh\alpha) \mathring{f}_m(\tau)\, d\tau$$

$$\mathring{f}_m(\tau) = \int_0^\infty f(\alpha) P^{-m}_{-1/2+i\tau}(\cosh\alpha) \sinh\alpha\, d\alpha \tag{4.2}$$

and setting $B(\tau) + (-1)^{m+1}\tau \tanh \pi\tau f_m{}^\circ(\tau) = A(\tau) \tanh\pi\tau$, we arrive at Equations

$$\int_0^\infty \tau A(\tau)[1 + g(\tau)] P^m_{-1/2+i\tau}(\cosh\alpha)\, d\tau = h(\alpha) \qquad (0 \leqslant \alpha < \alpha_0)$$

$$\int_0^\infty A(\tau) \tanh \pi\tau P^m_{-1/2+i\tau}(\cosh\alpha)\, d\tau = 0 \qquad (\alpha_0 < \alpha < \infty) \tag{4.3}$$

Here (4.4)

$$h(\alpha) = (-1)^{m+1} \int_0^\infty \mathring{f}_m(\tau) \tau^2 \tanh \pi\tau \coth \gamma\tau P^m_{-1/2+i\tau}(\cosh\alpha)\, d\tau, \quad g(\tau) = \frac{\sinh(\pi-\gamma)\tau}{\cosh \pi\tau \sinh \gamma\tau}$$

Equations (4.3) differ somewhat in structure from (1.1) and (1.2) and as a result the method of Section 1 must be modified. We set

$$A(\tau) = \int_0^{\alpha_0} \varphi(t) \sin \tau t \, dt, \quad \varphi(0) = 0 \tag{4.5}$$

and then Equation (4.3) will be satisfied; after the necessary transformations we arrive at the Fredholm integral equation (1.10) with the kernel

$$K(x, t) = G(t - x) - G(t + x) \tag{4.6}$$

and the right-hand side

$$\Phi(x) = \frac{2}{\pi} \int_0^x [2(\cosh x - \cosh \alpha)]^{-1/2} h(\alpha) \sinh \alpha \, d\alpha \tag{4.7}$$

The constants c_k which appear in $h(\alpha)$ (see (1.6)) can be found from the condition that $\partial u/\partial z$ is integrable over the area $z = 0$, $a < r < b$, which, after computations as before, reduces to (2.3).

Note that for the case of a half-space ($\gamma = \pi$), by means of dual equations we can derive a solution for mixed boundary-value problems of a slightly different type when a boundary condition of the first kind is specified on the ring $z = 0$, $a < r < b$ and a condition of the second kind is specified on the remainder of the boundary. An example of such a case, would be the electrostatic problem of a flat ring situated in an arbitrary external field. Mathematically this is equivalent to the problem of the equilibrium of an elastic half-space with the following boundary conditions: $z = 0$, $a < r < b$ the normal displacement is given, and on the remainder of the boundary the normal stress is given, the shear stresses at $z = 0$ being known. Particular examples of such a problem are provided by certain problems on stress concentration in an infinite body with internal and external circular cracks and also the contact problem for a hollow cylindrical (annular) die, which has been studied by many authors.

As a specific example of the elasticity problems we shall solve the problem of a half-space under torsion from an attached rigid annular die. Here the state of stress and strain is determined by the one function $u_\theta(r, z) \equiv u$, which satisfies Equation

$$\frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial u}{\partial r} - \frac{u}{r^2} + \frac{\partial^2 u}{\partial z^2} = 0, \qquad z > 0 \tag{4.8}$$

and boundary conditions

$$u|_{\beta=0} = \varepsilon r, \quad \alpha_0 < \alpha < \infty; \qquad \left.\frac{\partial u}{\partial \beta}\right|_{\beta=0} = 0, \quad 0 \leqslant \alpha < \alpha_0; \qquad \left.\frac{\partial u}{\partial \beta}\right|_{\beta=\pi} = 0 \tag{4.9}$$

where ϵ is the angle of rotation of the die.

If we seek a solution in the form

$$u = \varepsilon b \sqrt{\cosh \alpha + \cos \beta} \int_0^\infty B(\tau) \frac{\cosh(\pi - \beta)\tau}{\cosh \pi\tau} P^1_{-1/2+i\tau}(\cosh \alpha) \, d\tau \tag{4.10}$$

then the boundary conditions reduce to the dual integral equations

$$\begin{gathered} \int_0^\infty B(\tau) P^1_{-1/2+i\tau}(\cosh \alpha) \, d\tau = \frac{\tanh^{1/2} \alpha}{\sqrt{\cosh \alpha + 1}} \qquad (\alpha_0 < \alpha < \infty) \\ \int_0^\infty \tau B(\tau) \tanh \pi\tau \, P^1_{-1/2-i\tau}(\cosh \alpha) \, d\tau = 0 \qquad (0 \leqslant \alpha < \alpha_0) \end{gathered} \tag{4.11}$$

We expand the right-hand side of the first equation in a Mehler-Fock integral [6]

$$\frac{\tanh {}^1/_2\,\alpha}{\sqrt{\cosh\alpha+1}} = -2\sqrt{2}\int_0^\infty P^1_{-{}^1/_2+i\tau}(\cosh\alpha)\frac{d\tau}{\cosh\pi\tau} \tag{4.12}$$

Then, taking $B(\tau) + 2\sqrt{2}/\cosh\pi\tau = A(\tau)\tanh\pi\tau$ and taking into account Formulas

$$\int_0^\infty \frac{\tau\sinh\pi\tau}{\cosh^2\pi\tau} P_{-{}^1/_2+i\tau}(\cosh\alpha)\,d\tau = \frac{1}{2\pi\cosh^2{}^1/_2\,\alpha}, \qquad P^1_{-{}^1/_2+i\tau}(\cosh\alpha) = \frac{d}{d\alpha}P_{-{}^1/_2+i\tau}(\cosh\alpha) \tag{4.13}$$

we obtain the dual equations (4.3), where $m = 1$

$$g(\tau) = -\frac{1}{\cosh^2\pi\tau}, \qquad h(\alpha) = \frac{\sqrt{2}}{\pi}\frac{d}{d\alpha}\frac{1}{\cosh^2{}^1/_2\,\alpha} \tag{4.14}$$

Therefore, by making the substitution (4.5) we reduce the problem to the Fredholm equation (1.10) with the kernel (4.6) where

$$G(y) = -\frac{1}{\pi}\frac{{}^1/_2\,y}{\sinh{}^1/_2\,y}, \qquad \Phi(x) = \frac{2\sqrt{2}\,x}{\pi^2\cosh{}^1/_2\,x} + c\sinh{}^1/_2\,x \tag{4.15}$$

the constant c being determined from the condition $\varphi(\alpha_0) = 0$.

To obtain the complete solution we must express the angle ε im terms of the torque applied to the die

$$M = -G\int_a^b\int_0^{2\pi}\tau_0 r^2\,dr\,d\theta \qquad \tau_0 = \frac{\partial u}{\partial z}\Big|_{\beta=0} (\alpha_0<\alpha<\infty) \tag{4.16}$$

where G is the shear modulus. Taking into account the condition $\varphi(\alpha_0) = 0$ we can express the normal derivative in (4.16), after certain transformations, in the following form:

$$\tau_0 = \varepsilon(\cosh\alpha+1)^{3/2}\frac{\sqrt{2}}{\pi}\frac{d}{d\alpha}\left[\frac{1}{\cosh^2{}^1/_2\,\alpha} - \int_0^{\alpha_0}\varphi'(t)(\cosh\alpha-\cosh t)^{-1/2}\tan^{-1}\sqrt{\frac{\cosh t+1}{\cosh\alpha-\cosh t}}\,dt\right]$$

$$(\alpha_0<\alpha<\infty) \tag{4.17}$$

Carrying out the integration we find after a series of computations the required relation in the form of a quadrature

$$M = \frac{16}{3}G\varepsilon b^3\left\{\frac{\cosh\alpha_0+\cosh^2{}^1/_2\,\alpha_0}{2\cosh^3{}^1/_2\,\alpha_0} - \frac{3}{4\sqrt{2}\cosh{}^1/_2\,\alpha_0}\int_0^{\alpha_0}\varphi'(t)\frac{dt}{\cosh{}^1/_2\,t} + \right.$$

$$\left. + \frac{3}{4}\cosh\frac{\alpha_0}{2}\int_0^{\alpha_0}\left[\frac{\varphi(t)}{2} - \tanh\frac{t}{2}\,\varphi'(t)\right]\tan^{-1}\left(\frac{\cosh t+1}{\cosh\alpha-\cosh t}\right)^{1/2}\frac{\tanh{}^1/_2\,t\,dt}{\sqrt{\cosh\alpha_0-\cosh t}}\right\} \tag{4.18}$$

When $\alpha_0 = 0$, Formula (4.18) gives the familiar relation for a circular die.

In conclusion, we note that a number of other quantities can also be expressed directly in terms of the basic function $\varphi(t)$; for example, for the displacements of points on the circle $z = 0$, $0 \leq r < a$ we have Formula

$$u\,|_{\beta=0} = \varepsilon r + \varepsilon b\cosh^2{}^1/_2\,\alpha\sinh\alpha\int_\alpha^{\alpha_0}[\varphi'(t) - \coth t\,\varphi(t)](\cosh t-\cosh\alpha)^{-1/2}\frac{dt}{\sinh t} \tag{4.19}$$

$$(0\leqslant\alpha<\alpha_0)$$

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